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Time-Optimal Slewing of Flexible Spacecraft

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The time-optimal slewing problem of flexible spacecraft is considered. The system is discretized by the assumed modes method, and the problem is solved for a linearized model in reduced state space by parameter optimization. Optimality is verified by the Maximum Principle. The linear solution is further used to obtain time-optimal solutions for the nonlinear problem. Some interesting symmetric and asymptotic properties are shown to be possessed by both the linear and the nonlinear problems.

I. Introduction

IN recent years there has been considerable interest in control for flexible structures. It is becoming increasingly important to take into consideration the flexibility of structures in the design process of control systems for modern aerospace vehicles. Consequently, requirements for active vibration suppression are becoming part of attitude control system specifications for modern aircraft and spacecraft. Single-axis slew maneuvers of simple flexible spacecraft, consisting of a rigid hub and flexible appendages, have been investigated extensively by Turner,[1] Turner and Junkins,[2] Turner and Chun,[3] Juang et al.,[4] and Breakwell.[5] The approach was based on the linear quadratic regulator (LQR) theory, where the cost was a combination of control effort and system energy. Under this formulation, closed-loop control is possible, and the feedback gains can be obtained in closed form (for the linearized system). Lisowski and Hale[6] extended these results by freeing structural parameters in the optimization process, so that both structural parameters and active control torques can be determined to minimize a specific cost functional. Experimental results for LQR designs have been reported by Juang et al.[7] The measurements were compared with analytical predictions, and sufficient agreement has been obtained. Skaar et al.[8] studied a satellite reorientation problem using thrusters rather than internal momentum transfer devices. The control is on-off in open loop. The cost was the postmaneuver elastic energy, and optimal switching points were obtained to minimize it.

A problem of current interest is to include the transition time in the cost to accomplish fast slew maneuvers. Vander Velde and He[9] investigated a problem of this kind where the cost is a combination of fuel and time. The solution again is open-loop on-off control of the spacecraft thrusters. Thompson et al.[10] considered near time-optimal controls by minimizing a sum of the time and a certain measure of residual energy. The solution is open loop and is continuous with respect to time. Meirovitch and Sharony[11] and Quinn and Meirovitch[12] explored the idea of controlling the rigid body by time-optimal controls, while considering the elastic mode control as a modified LQR problem. In this formulation, the rigid-body time-optimal solution is a forcing (disturbance) term in the elastic equations. Closed-loop solutions are sought to suppress the induced vibrations.

The purpose of this work is to study the pure time-optimal slewing problem, where the cost is the transition time only. This problem is currently being investigated, and first results have been reported.[13,14] In Ref. 13 the two-point boundary-value problem is approached directly. This, in spite of the analytical integrability of subarcs, is still a computational burden. The problem in Ref. 14 is transformed into a set of nonlinear algebraic equations, using symmetric properties of the optimal solution. Numerical solutions are obtained by homotopy methods. A disadvantage of this approach is that the assumed symmetry restricts the method to rest-to-rest maneuvers. The discussion in both works[13,14] is restricted to linear modeling of the spacecraft, while it is clear that for very fast slewing maneuvers the nonlinear terms cannot be neglected.

The objectives of work are, therefore, three. First, to construct a simple algorithm to solve the linear time-optimal slewing problem. It is, however, desirable to develop a more general scheme than those employed in Refs. 13 and 14. The second objective is to study asymptotic properties of the solutions to further reduce the computational burden, especially for onboard applications. The third objective is to extend the results to include the system nonlinearities.

The following sections are organized accordingly. Section II formulates the problem in modal state space, obtained by the assumed-modes method. In Sec. III, a novel parameter optimization technique is presented to remove the need for a two-point boundary-value problem solver. Numerous computational results, primarily for rest-to-rest maneuvers, are presented in Sec. IV. Section V studies some interesting asymptotic properties of the solutions. The nonlinear slewing problem is studied in Sec. VI. This section presents numerical solutions obtained by a two-point boundary-value problem solver that employs a multiple-shooting algorithm and uses the linear solutions as initial guesses. In Sec. VII, some interesting symmetric properties are shown to be possessed by both the linear and the nonlinear problems (for rest-to-rest maneuvers). The conclusions are given in Sec. VIII, along with some recommendations for further study.

II. Problem Formulation

We consider the time-optimal single-axis rotation problem for a system consisting of a rigid hub with an elastic appendage attached to it. The appendage is modeled as an Euler-Bernoulli cantilevered beam. The system is controlled by a single actuator that exerts an external torque on the rigid hub.

A semi-discrete model for this problem can be obtained by the assumed modes method.[15] In modal space we have the linearized model

$$\ddot{\eta}(t) + \Omega^2\eta(t) = Gu(t) \tag{1}$$

where η is the generalized coordinate vector, Ω^2 is a diagonal matrix with entries ω_i^2, u is the control (scalar) function, and G is the control influence vector with nonzero entries g_i.

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In the nonlinear model, the "centrifugal" stiffness is included,[1,2] leading to the model

$$\ddot{\eta}(t) + \Omega^2\eta(t) = Gu(t) - \left[G^T\dot{\eta}(t)\right]^2 L\eta(t) \tag{2}$$

where $(G^T\dot{\eta})^2L$ is the centrifugal stiffness matrix.

We shall transform the model to state space by using

$$x = [x_{-1},x_0,x_1,x_2,\ldots,x_{2k}]^T, \quad x_{2i-1} = \eta_i, \quad x_{2i} = \dot{\eta}_i$$
$$i = 0,1,\ldots,k$$

The linear state space model is

$$\dot{x}(t) = Ax(t) + bu(t) \tag{3}$$

where

$$A = \begin{bmatrix} 0 & 1 & & & & & \\ 0 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & -\omega_1^2 & 0 & & & \\ & & & & \vdots\ \vdots & & \\ & & & & & 0 & 1 \\ & & & & & -\omega_k^2 & 0 \end{bmatrix}$$

$$b = [0 \quad g_0 \quad 0 \quad g_1 \quad \ldots \quad 0 \quad g_k]^T$$

We assume $0<\omega_1<\ldots<\omega_k$. Note that k is the number of flexible modes, and $n=2k+2$ is the dimension of the system.

The nonlinear model is

$$\dot{x}(t) = Ax(t) + bu(t) - T_2^T\left[G^TT_2x(t)\right]^2 LT_1x(t)$$
$$\equiv f\left[x(t),u(t)\right] \tag{4}$$

where T_1 and T_2 are transformation matrices defined by $\eta = T_1x$, $\dot{\eta} = T_2x$.

The time-optimal problem is to find a measurable function u that drives, in minimal time, the state x from a given value $x(0)=x^0$ to a target set $x(t_f) \in \Theta$, subject to $|u(t)| \le u_M$, $0 \le t \le t_f$, where u_M is the maximum admissible torque.

III. Method of Solution

Consider the linear state equations (3). The system is marginally stable and controllable.[16] In the terminology of Lasalle, the system is also normal as a result of the one-dimensionality of the control space.[17] Existence and uniqueness of the optimal solution are guaranteed, and the Maximum Principle is a sufficient as well as a necessary condition for it. Singular subarcs can also be ruled out by normality.[18]

Let $\tilde{u}$ be the optimal control. The Maximum Principle asserts the existence of a nontrivial solution $\lambda(t)$ for the adjoint equation

$$\dot{\lambda}(t) = A^T\lambda(t) \tag{5}$$

where $\lambda(t_f)$ is orthogonal to Θ, and

$$\tilde{u}(t) = -u_M\, \mathrm{sgn}\left[\lambda^T(t)b\right] \tag{6}$$

Thus, the optimal control is bang-bang and governed by a switching function $\lambda^T(t)b$. The number of switching points is, however, unknown, and, unlike the case when A has real distinct eigenvalues, there is no upper limit for it. At each switching point the switching function must vanish.

Consider rest-to-rest slewing to a single point x^f. The solution to the problem can be obtained by parameter optimization. We formulate the following problem. Let $\{t_1,t_2,\ldots,t_m,t_f\}$, i.e., m switching points and the final time, be free parameters. We minimize the cost

$$J = t_f \tag{7}$$

subject to

$$u_M h_i(t_f) + 2u_M \sum_{j=1}^{m}(-1)^j h_i(t_f - t_j) = x_i^f - x_i^0$$
$$i = -1,0,1,\ldots,2k \tag{8}$$

where $h_i(t)$ are simple step-response functions. By the superposition principle, we consider the bang-bang control function as a linear combination of $m+1$ step functions. The first step starts at $t=0$, with amplitude u_m, and the rest take place at the switching points, with alternating signs, and with amplitude $2u_M$.

For the rigid body,

$$h_{-1}(t) = 0.5(g_0)t^2 \tag{9a}$$

$$h_0(t) = (g_0)t \tag{9b}$$

whereas for the elastic modes,

$$h_{2i-1}(t) = \omega_i^{-2}(g_i)(1-\cos\omega_i t) \tag{9c}$$

$$h_{2i}(t) = \omega_i^{-1}(g_i)\sin\omega_i t \tag{9d}$$

Notice that both the constraints and the cost are analytically differentiable with respect to each parameter. This fact is very important in parameter optimization, since gradient methods work with sensitivity derivatives of the cost and the constraints, and these derivatives are evaluated, in general, at each iteration step.

Since the number of switchings is not known, and for any given number of parameters an optimal set may exist, we need to verify that the solution obtained via parameter optimization is, indeed, the optimal solution. Recall that the optimal control problem is guaranteed to have a unique solution.

From Eq. (5) the adjoint vector $\lambda(t)$ is given by

$$\lambda(t) = e^{-A^Tt}\lambda(0) \tag{10}$$

Hence the switching function is

$$b^T\lambda(t) = b^Te^{-A^Tt}\lambda(0) = 0 \tag{11}$$

For each switching point t_i, $i=1,2,\ldots,m$, we obtain

$$b^Te^{-A^Tt_i}\lambda(0) = 0 \tag{12}$$

Construct the $m \times n$ matrix P:

$$P = \begin{bmatrix} b^Te^{-A^T}t_1 \\ b^Te^{-A^T}t_2 \\ \vdots \\ b^Te^{-A^T}t_m \end{bmatrix} \tag{13}$$

to obtain

$$P\lambda(0) = 0 \tag{14}$$

We make the following observations:

1) The $\lambda(0)$ is in the null space of P.

2) The nullity of P satisfies dim (null P) $= n -$ rank P.

3) By the Maximum Principle there exists a nontrivial solution to the homogeneous adjoint equations. Therefore, if the

solution obtained by parameter optimization is the optimal solution, the null P is not empty.

4) If the nullity is greater than 1, we may be able to use the transversality condition for further reducing this subspace by intersecting the two subspaces.

5) When $\lambda(0)$ is known, the Maximum Principle can be verified by directly evaluating $\lambda(t)$.

6) We can use $\lambda(0)$ of the linear problem as a starting point (along with the switching points) for the two-point boundary-value problem associated with the nonlinear model.

IV. Computational Results

The structural parameters are

hub moment of inertia = 981 (slug-ft^2)
beam rigidity (EI) = 3.49×10^4 (lbf-ft^2)
beam length (L) = 30 (ft)
beam mass distribution = 0.109 (slug/ft)
maximum torque = 221.4 (lbf-ft)

The values are appropriate for an aluminum beam that is 30 ft long and has a rectangular cross section with height 1 ft and thickness 0.25 in. These are representative values for a solar panel.[19]

A complete set of admissible functions is given by

$$\Phi_j(\xi) = \left(\frac{\xi}{L}\right)^{j+1}, \qquad j = 1,2,\ldots \tag{15}$$

The natural frequencies, as obtained by the assumed modes method,[15] are presented in Table 1. Five modes will be used to discretize the system. We consider rest-to-rest maneuvers

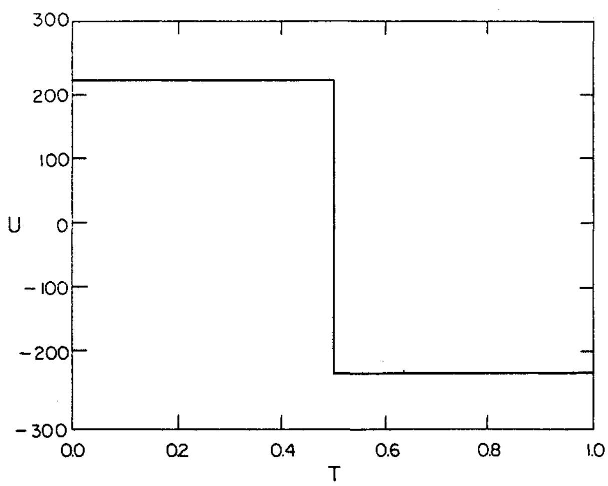

Fig. 1 Rigid body—rest to rest.

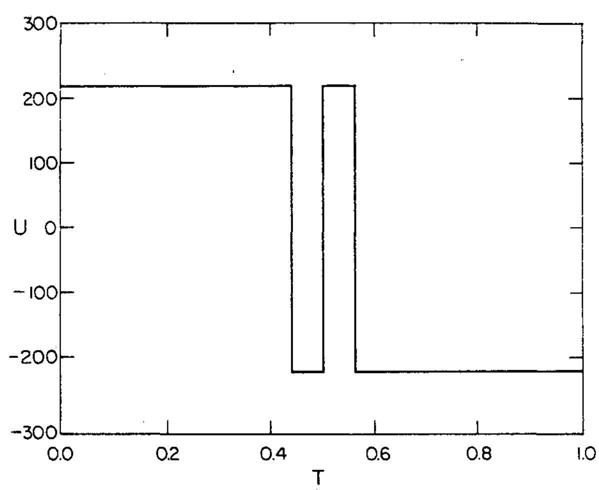

Fig. 2 One elastic mode—rest to rest.

Table 1 Natural frequencies, 1/s

Dimension of approximation	ω_0	ω_1	ω_2	ω_3	ω_4
1	0	—	—	—	—
2	0	3.857	—	—	—
3	0	3.089	22.135	—	—
4	0	3.081	14.147	74.380	—
5	0	3.080	14.102	39.864	56.20

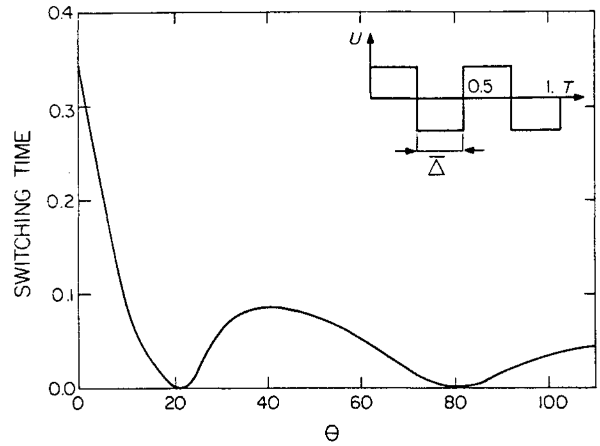

Fig. 3 Switching points vs slew angle.

where the system is to be driven to $x(t_f)=0$ from $x(0) = [-\theta,0,0,0,\ldots]^T$.

The time-optimal solution for the rigid-body problem is very well known. The optimal control function is shown in Fig. 1 (time is normalized by t_f). There is a single switching at the midpoint, and the terminal time is

$$t_f = 2\sqrt{\frac{\theta}{g_0 u_M}} \tag{16}$$

Figure 2 presents, in normalized time, a solution for the problem ($\theta = 30$ deg), where the first elastic mode (with the fundamental frequency $\omega_1 = \omega$) is taken into account. Solutions for the parameter optimization problem have been obtained by an optimization program that employs an iterative procedure based on a variable-metric gradient-projection algorithm.[20] It turns out that there are three switching points: the original one at the midpoint and two others symmetrically located before and after the middle point. The terminal time is 5.0 s, whereas for the rigid-body problem the slewing time is 4.9 s. The switching function has been obtained by the null space technique to verify the optimality of the solution. It is noted that for some initial estimates of the time parameters the parameter optimization procedure found other solutions, with three switching points, that failed to satisfy the Maximum Principle.

As we vary the slew angle, the switching at the midpoint (in normalized time) remains fixed, but the other two move as shown in Fig. 3. Figure 4 shows the slewing time vs the slew angle for the problem. Notice that at certain points the solution coincides with the rigid-body solution. From Fig. 3 we conclude that at these points there is, indeed, only one switching point (i.e., the rigid-body solution). To explain this phenomenon one may evaluate the response of the one-mode elastic equation to the rigid-body optimal control function to obtain

$$x_1(t_f) = -2(g_1 u_M)\omega^{-2}\cos\left(\frac{\omega t_f}{2}\right)\sin^2\left(\frac{\omega t_f}{4}\right) \tag{17a}$$

$$x_2(t_f) = -2(g_1 u_M)\omega^{-1}\sin\left(\frac{\omega t_f}{2}\right)\sin^2\left(\frac{\omega t_f}{4}\right) \tag{17b}$$

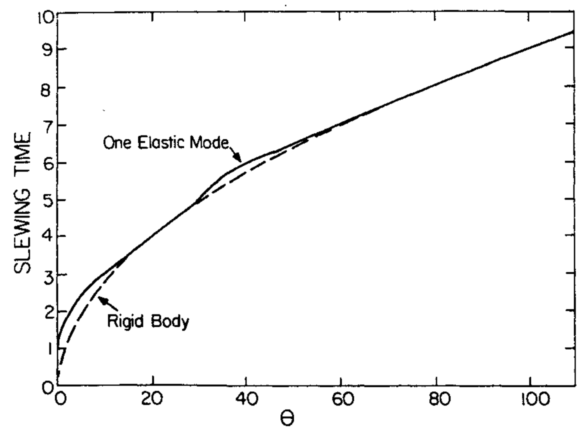

Fig. 4 Slewing time vs slew angle.

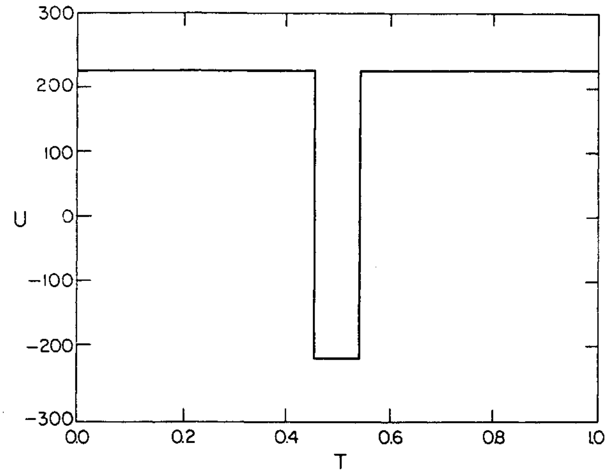

Fig. 5 One elastic mode—spinup.

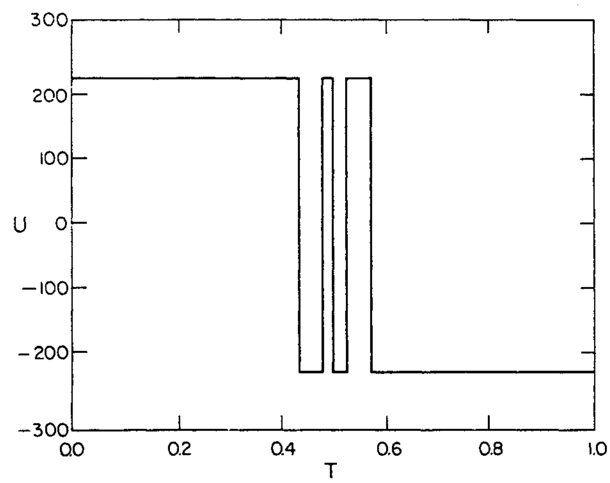

Fig. 6 Two elastic modes—rest to rest.

Hence, in agreement with the numerical results, for countably many points that satisfy

$$\frac{\omega t_f}{4} = p\pi, \qquad p = 0,1,2,\ldots \tag{18}$$

the elastic boundary conditions are satisfied by the rigid-body optimal control so that the solution contains only one switching point. The matrix P in Eq. (13) is therefore $[1\times 4]$ with nullity 3, and λ is not unique. This is an example for a nonunique λ that determines a unique optimal control. As pointed out by Hermes and Lasalle,[17] *uniqueness of the control and its being determined uniquely by the necessary condition have nothing to do with the uniqueness of the direction* λ.

For most cases—when Eq. (18) is not satisfied—the optimal control contains three switching points. The matrix P is a $[3\times 4]$ matrix with one-dimensional null space. The initial adjoint vector is, therefore, uniquely determined by the switching structure.

To demonstrate a different situation, we consider a minimal time spinup maneuver. The problem is to drive the system from 0 to $x(t_f) = [\theta,\Omega,0,0]$, where Ω is given, θ is free, and the cost to be minimized is, again, the terminal time. Notice that the target set Θ is now the entire x_1 axis rather than a single point. The solution for the rigid-body equivalent problem is, of course, a constant maximal control torque. The solution for the elastic case (one flexible mode) is shown in Fig. 5. There are two switching points, and the symmetry is preserved. However, P is a $[2\times 4]$ matrix of rank 2 (and hence nullity 2). Based on the transversality conditions, $\lambda(t_f)$ is orthogonal to Θ; hence $\lambda_1(t)$ (which is constant by the adjoint equations) is zero. This defines a three-dimensional subspace, the intersection of which with null (P) yields the desired $\lambda(0)$, and the Maximum Principle can be verified.

We shall return now to rest-to-rest maneuvers to include the first two elastic modes in our control design. Figure 6 presents the solution for our test case ($\theta = 30$). The total time is only 0.004 s more than the one elastic mode case. The number of switching points is five, and the symmetry is preserved.

Notice that three of the switching points seem to be small perturbations of the one flexible mode case, and the two new switches take place near the midpoint. An interesting question is whether the number of switching points for rest-to-rest maneuvers with two flexible modes is always five or less (and therefore is consistent with the real eigenvalue cases[18]). To answer this question a different slew angle is considered. Figure 7 presents the optimal five switching points for $\theta = 10$; the terminal time is very close to the one flexible mode solution (Fig. 4), but the Maximum Principle is not satisfied! Further investigation reveals a different solution made of seven switching points (Fig. 8) that satisfies the Maximum Principle and yields a slightly better transition time. It is noted that the optimal solution is still a perturbation to the one flexible mode solution, with two more pulses created by the four new switches.

The residual energies (the energy stored in the uncontrolled modes) for the test case ($\theta = 30$) have been calculated using five modes. The energy is 5.08 lbf-ft for the rigid-body solution, 0.07 lbf-ft for the one flexible mode, and 0.009 lbf-ft for the two flexible modes. Therefore, for this case, the error

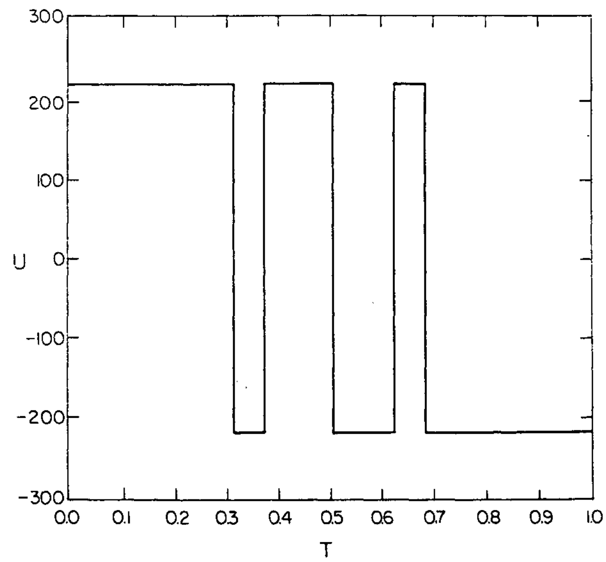

Fig. 7 Two elastic modes—rest to rest.

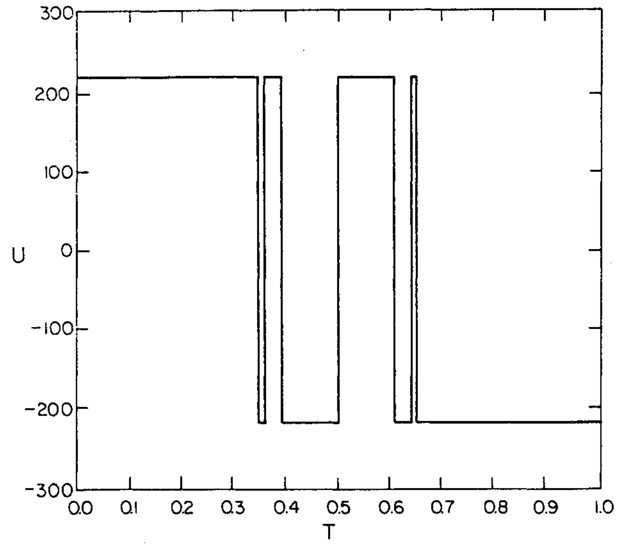

Fig. 8 Two elastic modes—rest to rest.

Table 2 Terminal time, s

	EI (nom)	0.1 EI	0.02 EI
Rigid-body control	4.90	4.90	4.90
One-mode control	5.000	6.69	10.14
Two-mode control	5.004	6.71	10.41

Table 3 Residual energy, lb-ft

	EI	0.1 EI	0.02 EI
Rigid-body control	5.08	293.5	136.2
One-mode control	0.07	0.243	4.047
Two-mode control	0.009	0.002	0.032

obtained by controlling only the fundamental mode is relatively very small.

Two more cases have been studied, in which we consider the beam rigidity EI to be 10% and 2% of its nominal value. Tables 2 and 3 summarize the results for the cost (the terminal time) and the residual energy. It turns out that the first flexible mode is still dominant in determining the terminal time. However, the residual energy may not be negligible under one flexible mode control; hence more controlled modes are required.

The symmetry of the switching-point structure has been preserved in all cases. This fact is true as long as the system is undamped, but when structural damping is included in the model, the switchings are no longer symmetric with respect to the midpoint.

V. Asymptotic Behavior of the Solution

In this section we consider rest-to-rest maneuvers with one elastic mode only, and we investigate the behavior of the solution as the flexibility of the system is changed by varying ω. The basic structure of the solution, i.e., switching points at $\{t_f/2-\Delta,\ t_f/2,\ t_f/2+\Delta\}$, is not affected by ω. The value of Δ, however, varies with ω as shown in Fig. 9 (normalized by t_f), for three different slew angles. There are two asymptotes. For high ω, Δ approaches zero, i.e., the rigid-body solution, whereas for low ω, $\bar{\Delta}\equiv\Delta/t_f$ approaches $(1/\sqrt{8})$. To understand this behavior, consider a zeroth-order approximation for the elastic equation. As $\omega\rightarrow 0$, the step response of the elastic deflection, Eq. (9c), approaches $h_1(\tau)\rightarrow(g_1u_M)\tau^2/2$, and the terminal value is by superposition

$$x_1(t_f)\rightarrow(g_1u_M)(\tfrac{1}{4}-2\bar{\Delta}^2)t_f^2 \tag{19}$$

Since the system is driven to the origin, it is clear that $\bar{\Delta}\rightarrow(1/\sqrt{8})$. The rigid-body response is for any ω

$$x_{-1}(t_f) = -\theta + (g_0u_M)(\tfrac{1}{4}-2\bar{\Delta}^2)t_f^2 \tag{20}$$

Therefore, $t_f\rightarrow\infty$ as $\omega\rightarrow 0$. (For $\omega=0$, the system is not controllable.[16])

The behavior of the terminal time with ω is shown in Fig. 10 on a logarithmic scale. Again, there are two asymptotes: for high ω the terminal time is the rigid-body problem optimal time t_f, whereas for low ω the asymptotic line is

$$\omega\times t_f^2 = \text{const} \tag{21}$$

To explain this behavior, we make a first-order approximation for the elastic step response function, Eq. (9c),

$$\omega^{-2}(g_1)(1-\cos\omega\tau)\sim(g_1)\left(\frac{\tau^2}{2}-\frac{\omega^2\tau^4}{24}\right) \tag{22}$$

Under this approximation the terminal elastic deflection is for $\omega\rightarrow 0$

$$x_1(t_f)\rightarrow\theta\frac{g_1}{g_0}-(g_1u_M)\frac{\omega^2t_f^4}{384} \tag{23}$$

Hence,

$$\omega\times t_f^2\rightarrow\sqrt{\frac{384\theta}{(g_0u_M)}} \tag{24}$$

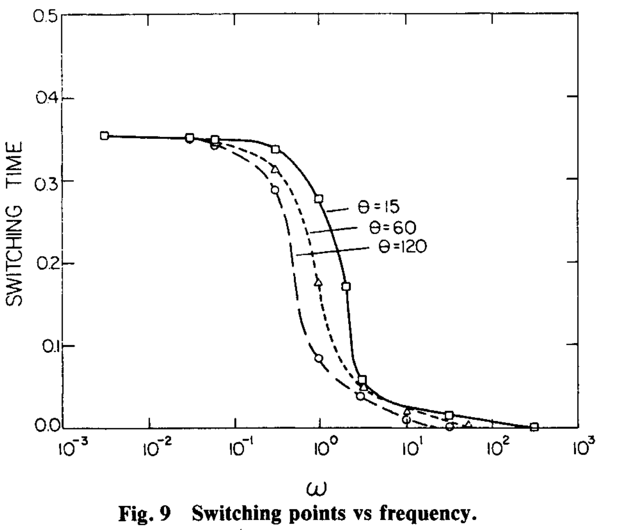

Fig. 9 Switching points vs frequency.

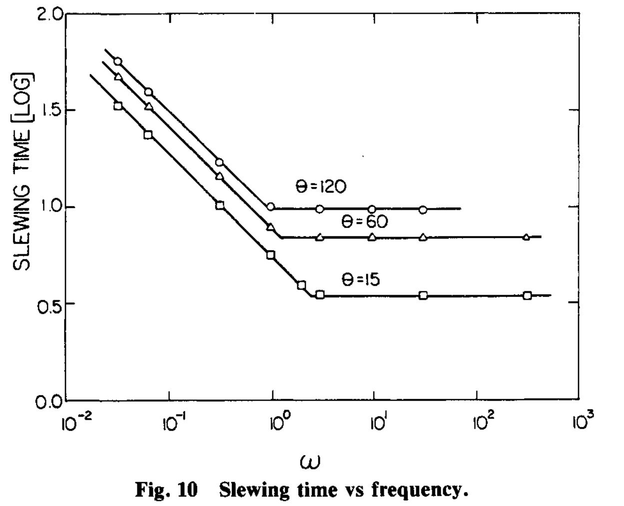

Fig. 10 Slewing time vs frequency.

which is in agreement with the numerical results. The break point of the two asymptotes is, by a simple calculation,

$$\omega_b = \frac{\sqrt{96}}{t_{f_r}} \tag{25}$$

Note that by Eq. (16) t_{f_r} depends on system parameters g_0 and u_M and on the initial value θ. An interesting fact that comes up from the optimization results and could not be predicted by the asymptotic analysis alone is that the asymptotes provide a very good estimate for t_f over the entire domain of ω.

It is also important to note that the same asymptotic behavior is obtained for a fixed ω but $t_f \to 0$, where the product $\omega\tau$ is, again, a small quantity. This can be useful for approximating solutions for small slew angles or for high admissible torques (see examples in Ref. 21).

VI. Nonlinear Effects

Consider the nonlinear model Eq. (4). The Maximum Principle is a necessary condition for optimality. Define the usual variational Hamiltonian

$$H(x,\lambda,u) = 1 + \lambda^T f(x,u) \tag{26}$$

The Maximum Principle requires that[18]

$$\dot{\lambda}_i = -\frac{\partial H}{\partial x_i}, \qquad i = -1,0,1,2,\ldots,2k \tag{27}$$

$$H[x(t_f),\lambda(t_f),u(t_f)] = 0 \tag{28}$$

and

$$\bar{u}(t) = -u_M \operatorname{sgn}[\lambda^T(t)b] \tag{29}$$

The solution, therefore, is either bang-bang or singular [i.e., subarcs, along which the switching function $\lambda^T(t)b$ vanishes, are part of the trajectory]. The previously described parameter optimization cannot be applied directly to this problem, and a two-point boundary-value problem solver is, therefore, required. If, however, a good initial guess is available, for the control function as well as for the initial boundary condition, then the computational effort might become minimal. A natural idea is to use the linear results as a starting point for the algorithm. (A similar concept has been applied in Refs. 1 and 2 for minimizing a quadratic performance index of the same system.)

Solutions for the nonlinear problem have been obtained by a multiple-shooting algorithm.[22] For our nominal numerical values, the differences in the trajectory between the linear and the nonlinear models are very small. The solution is nonsingular, the number of switchings is the same, and the symmetry is preserved. The solution is, in fact, a perturbation of the linear case. To further investigate the nonlinear effects, we increase the maximal admissible torque by the factor 10^p where $p = 0,1,\ldots,6$. The angular velocities, and hence the nonlinear effects, should increase accordingly. Numerical results for one flexible mode indicate that the nonsingular switching-point structure is preserved. The distance between switches $\bar{\Delta}$, as a function of p, is shown in Fig. 11 for both linear and nonlinear cases. The linear solution approaches its asymptotic line as before. The nonlinear solution has a smaller asymptotic value. The transition time is shown in Fig. 12 on a logarithmic scale. As Eq. (23) predicts, the slope of the linear solution is $-\frac{1}{4}$. The stiffer nonlinear case has a smaller transition time.

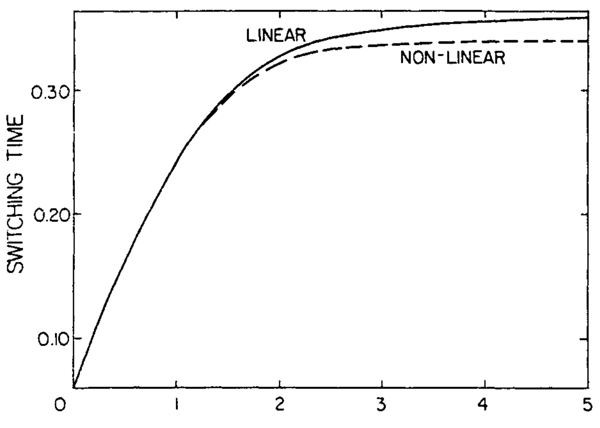

Fig. 11 Switching points vs torque.

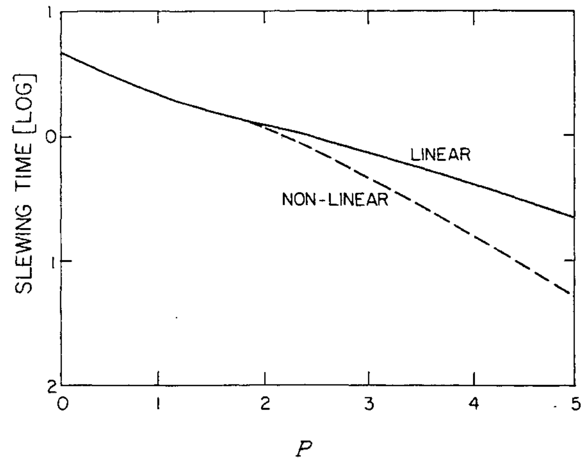

Fig. 12 Slewing time vs torque.

VII. Symmetric Properties of Rest-to-Rest Maneuvers

We have found numerically that for the undamped system, linear or nonlinear, the time-optimal solution for rest-to-rest maneuvers possesses the symmetry

$$u(t) = -u(t_f - t) \tag{30}$$

To justify this analytically, we shall reconsider the problem in the original modal space. The governing equation is therefore

$$\ddot{\eta}(t) + \Omega^2\eta(t) = Gu(t)$$

The optimal rest-to-rest maneuvering problem was to find a measurable function $u(\cdot):[0,t_f] \to R$ that, subject to Eq. (1), will satisfy the boundary conditions

$$\eta^T(0) = [-\theta,0,0,\ldots], \quad \dot{\eta}^T(0) = [0,0,0,\ldots]$$

$$\eta^T(t_f) = [0,0,0,\ldots], \quad \dot{\eta}^T(t_f) = [0,0,0,\ldots]$$

while minimizing t_f. Let $\tilde{u}(\cdot):[t_0,\tilde{t}_f] \to R$ be the optimal control function, where $\tilde{t}_f$ is the minimum time. We shall demonstrate that the symmetry in Eq. (30) is true. This will be done in three steps.

1) Define a new independent variable $\tau \equiv \tilde{t}_f - t$. Since

$$-\frac{d\eta(\tau)}{d\tau} = \frac{d\eta(t)}{dt}, \qquad \frac{d^2\eta(\tau)}{d\tau^2} = \frac{d^2\eta(t)}{dt^2}$$

and since Ω^2 and G are constants, we obtain from Eq. (1)

$$\frac{d^2\eta(\tau)}{d\tau^2} + \Omega^2\eta(t) = Gu(\tau) \tag{31}$$

The optimal control problem can now be reformulated to find $u(\cdot):[0,\tau_f]\to R$ that, subject to Eq. (31), will satisfy the boundary conditions

$$\eta^T(\tau)|_{\tau=0} = [0,0,\ldots], \quad -\frac{d\eta^T(\tau)}{d\tau}|_{\tau=0} = [0,0,\ldots]$$

$$\eta^T(\tau)|_{\tau=\tau_f} = [-\theta,0,0,\ldots], \quad -\frac{d\eta^T(\tau)}{d\tau}|_{\tau=\tau_f} = [0,0,\ldots]$$

while minimizing τ_f.

Let $\hat{u}(\cdot):[0,\hat{\tau}_f]\to R$ be the optimal control function, where $\hat{\tau}_f$ is the minimum value obtained for τ_f; then clearly

$$\hat{\tau}_f = \tilde{t}_f$$

$$\hat{u}\left[\tau(t)\right] = \tilde{u}(t) \tag{32}$$

since these are solutions to the same optimal control problem.

2) Consider again the original formulation. We make another change of variables by defining a new dependent variable

$$\zeta^T \equiv \eta^T + [\theta,0,0,\ldots]$$

(i.e., shifting the first component of η by the constant value θ). Due to the fact that the first column of Ω^2 is zero, the governing equation is still

$$\ddot{\zeta}(t) + \Omega^2\zeta(t) = Gu(t) \tag{33}$$

but the boundary conditions for our problem become

$$\zeta^T(0) = [0,0,0,\ldots], \quad \dot{\zeta}^T(0) = [0,0,0,\ldots]$$

$$\zeta^T(t_f) = [\theta,0,0,\ldots], \quad \dot{\zeta}^T(t_f) = [0,0,0,\ldots]$$

Clearly the time-optimal solution to this problem remains $\tilde{u}(\cdot):[t_0,\tilde{t}_f]\to R$.

3) Comparing the problem formulation just obtained with the one obtained in the previous step, we observe that we have identical governing equations (31) and (33), with zero initial conditions, but with opposite signs for the terminal conditions! Since the problem is linear, we conclude that the optimal control functions should have the same magnitude with opposite signs; thus

$$\hat{u}(\cdot) = -\tilde{u}(\cdot) \tag{34}$$

The physical meaning of this fact is that clockwise and counterclockwise rest-to-rest maneuvers have opposite signs for the controls. From Eq. (34),

$$\hat{u}\left[\tau(t)\right] = -\tilde{u}\left[\tau(t)\right] \tag{35}$$

Hence, from Eq. (32),

$$\tilde{u}(t) = -\tilde{u}\left[\tau(t)\right] = -\tilde{u}(\tilde{t}_f - t) \tag{36}$$

which is what we sought.

We shall make the following notes:

1) Although more complicated, the same ideas can be applied to the nonlinear case, since the essence of the proof is the fact that Eqs. (1) and (2) are invariant with respect to the time direction. In Ref. 23 the authors indicated that the nonlinear system did not possess this symmetry. It was found, however, that the numerical results were not fully converged and that converged solutions are symmetric.

2) Introducing damping into the system breaks this symmetry. A term $\Pi\dot{\eta}$ is introduced into the governing equation where Π is a constant matrix. This term, however, is not invariant with respect to time direction!

3) The previous arguments hold for some other cost functionals such as LQR with diagonal weighting matrices. This can be confirmed by the symmetric results presented in Refs. 3–5. Case 4L in Ref. 2 seems to contradict this assertion; however, the numerical results for this case are incorrect.[24]

VIII. Conclusions

The time-optimal maneuvering problem for a linearly elastic spacecraft has been transformed into a problem in nonlinear programming. A method has been developed to verify the optimality of the computed solutions. An asymptotic analysis affords a good approximation for the solution, provided that the system's elastic response is dominated by its fundamental frequency. Solutions based on a nonlinear model can be generated from the linear results. Interesting and useful symmetry properties have been established for both the linear and the nonlinear models. For the example studied, solutions based on higher-order models produce decreasing residual energy (the energy remaining after the maneuver) when applied to a "truth" model. The approach offers possibilties for use in combined design/control studies by including system parameters in the nonlinear programming formulation.

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